DISCRETE TRANSPARENT BOUNDARY CONDITIONS FOR WIDE ANGLE PARABOLIC EQUATIONS IN UNDERWATER ACOUSTICS

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This paper is concerned with transparent boundary conditions (TBCs) for wide angle "parabolic" equations (WAPEs) in underwater acoustics (assuming cylindrical symmetry). Existing discretizations of these TBCs introduce slight numerical reflections at this artificial boundary and also render the overall Crank–Nicolson finite difference method only conditionally stable. Here, a novel discrete TBC is derived from the fully discretized whole–space problem that is reflection–free and yields an unconditionally stable scheme.

A much more detailed version of this article will be published elsewhere [3].

1. INTRODUCTION

This paper is concerned with a finite difference discretization of *standard* and *wide angle "parabolic" equations* (see e.g. [9]). These models appear as one–way approximations to the Helmholtz equation in cylindrical coordinates with azimuthal symmetry. In particular we will discuss the discretization of transparent bottom boundary conditions.

In oceanography one wants to calculate the underwater acoustic pressure $p(z,r)$ emerging from a time–harmonic point source located in the water at $(z_s,0)$. Here, $r>0$ denotes the radial range variable and $0<z<z_b$ the depth variable. The water surface is at $z=0$, and the (horizontal) sea bottom at $z=z_b$. We denote the local sound speed by $c(z,r)$, the density by $\rho(z,r)$, and the attenuation by $\alpha(z,r)\geq 0$. $n(z,r)=c_0/c(z,r)$ is the refractive index, with a reference sound speed c_0 The reference wave number is $k_0=2\pi f/c_0$, where f denotes the (usually low) frequency of the emitted sound.

The pressure satisfies the Helmholtz equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial p}{\partial r}\right)+\rho\,\frac{\partial}{\partial z}\left(\rho^{-1}\,\frac{\partial p}{\partial z}\right)+k_0^2N^2p=0,\qquad r>0, \tag{1}$$

with the complex refractive index $N(z,r)=n(z,r)+i\alpha(z,r)/k_0$. In the far field approximation ($k_0r\gg 1$) the (complex valued) outgoing acoustic field $\psi(z,r)=\sqrt{k_0r}\,p(z,r)\,e^{-ik_0r}$

satisfies the *one–way Helmholtz equation*:

$$\psi_r = ik_0\big(\sqrt{1-L}-1\big)\psi, \qquad r>0. \tag{2}$$

Here, $\sqrt{1-L}$ is a pseudo–differential operator, and L the Schrödinger operator

$$L = -k_0^{-2}\rho\,\partial_z(\rho^{-1}\partial_z) + V(z,r), \qquad V(z,r) = 1 - N^2(z,r). \tag{3}$$

"Parabolic" approximations of (2) consist in formally approximating the pseudo–differential operator $\sqrt{1-L}$ by rational functions of L [4, 7, 9]. The linear approximation of $\sqrt{1-\lambda}$ by $1-\frac{\lambda}{2}$ gives the narrow angle or *standard "parabolic" equation* (SPE)

$$\psi_r = -\frac{ik_0}{2}\,L\psi, \qquad r>0. \tag{4}$$

This equation is a reasonable description of waves with a propagation direction within about 15° of the horizontal. Rational approximations of the form $(1-\lambda)^{\frac{1}{2}} \approx f(\lambda) = (p_0 - p_1\lambda)/(1-q_1\lambda)$ with real p_0, p_1, q_1 yield the *wide angle "parabolic" equations* (WAPE)

$$\psi_r = ik_0\left(\frac{p_0 - p_1 L}{1-q_1 L} - 1\right)\psi, \qquad r>0. \tag{5}$$

In the sequel we will repeatedly require $f'(0) = p_0 q_1 - p_1 < 0$. With the choice $p_0 = 1$, $p_1 = \frac{3}{4}$, $q_1 = \frac{1}{4}$ ((1,1)–Padé approximant of $(1-\lambda)^{\frac{1}{2}}$) one obtains the WAPE of Claerbout.

In this article we shall focus on boundary conditions (BCs) for the SPE (4) and the WAPE (5). At the water surface one usually employs a Dirichlet BC : $\psi(z=0,r) = 0$. At the sea bottom the wave propagation in the water has to be coupled to the wave propagation in the sediments of the bottom. The bottom will be modeled as the homogeneous half–space region $z > z_b$ with constant parameters c_b, ρ_b, and α_b. Throughout this paper we will use a fluid model for the bottom by assuming that (5) also holds for $z > z_b$.

In practical simulations one is only interested in the acoustic field $\psi(z,r)$ in the water, i.e. for $0 < z < z_b$. While the physical problem is posed on the unbounded z–interval $(0,\infty)$, one wishes to restrict the computational domain in the z–direction by introducing an artificial boundary at or below the sea bottom. Until recently, the standard strategy was to introduce rather thick absorbing layers below the sea bottom and then to limit the z–range by again imposing a Dirichlet BC [4]. This, of course, artificially changes the model and it increases the computational costs significantly. In [11] Papadakis derived *impedance BCs* or *transparent boundary conditions* (TBCs) for the SPE and the WAPE: complementing the WAPE (5) with a TBC at z_b allows to recover — on the finite computational domain $(0, z_b)$ — the exact half–space solution on $0 < z < \infty$. As the SPE is a Schrödinger equation, similar strategies have been developed independently for quantum mechanical applications (see e.g. [2] and the references therein).

While TBCs fully solve the problem of cutting off the z–domain for the analytical equation, all available numerical discretizations suffer from reduced accuracy (in comparison to the discretized half–space problem) and render the overall numerical scheme only conditionally stable [10, 12]. The object of this paper is to construct *discrete transparent boundary conditions* (DTBCs) for a Crank–Nicolson finite difference discretization of the WAPE such that the overall scheme is unconditionally stable and as accurate as the discretized half–space problem.

2. TRANSPARENT BOUNDARY CONDITIONS

As the density is typically discontinuous at the water–bottom interface ($z = z_b$), one requires continuity of the pressure and the normal particle velocity (*matching conditions*):

$$\psi(z_{b}-, r) = \psi(z_{b}+, r), \quad \psi_z(z_{b}-, r)/\rho_w = \psi_z(z_{b}+, r)/\rho_b, \tag{6}$$

where $\rho_w = \rho(z_b-, r)$ and ρ_b denotes the constant density of the bottom.

With (6) one can easily derive an estimate for the L^2–decay of solutions to the WAPE (5), $z > 0$. We assume $\rho = \rho(z)$ and apply the operator $1 - q_1 L$ to (5):

$$\left[1-q_1 V+q_1 k_0^{-2}\rho\,\partial_z(\rho^{-1}\partial_z)\right]\psi_r = ik_0\left[p_0-1-(p_1-q_1)V+(p_1-q_1)k_0^{-2}\rho\,\partial_z(\rho^{-1}\partial_z)\right]\psi. \tag{7}$$

A standard procedure gives for the weighted L^2–norm $\|\psi(.,r)\|^2 = \int_0^\infty |\psi(z,r)|^2\,\rho^{-1}(z)\,dz$

$$\partial_r\|\psi(.,r)\|^2 = -2\,C_1 \int_0^\infty \alpha\,\frac{c_0}{c}\,\left|\tilde{\partial}_r\psi\right|^2 \rho^{-1}\,dz, \quad C_1 = \frac{2\,(p_1-q_1)^2}{p_1-p_0q_1}, \tag{8}$$

In the dissipation–free case ($\alpha \equiv 0$) $\|\psi(.,r)\|$ is conserved and for $\alpha > 0$ it decays.

Now we sketch the derivation of the TBC for the WAPE. We assume that the initial data $\psi^I = \psi(z, 0)$ is supported in $0 < z < z_b$ and that all physical parameters are constant for $z > z_b$. The TBC for the SPE (or Schrödinger equation) was derived in [2, 10, 11]:

$$\psi(z_b, r) = -(2\pi k_0)^{-\frac{1}{2}}\,e^{\frac{\pi}{4}i}\,\frac{\rho_b}{\rho_w}\int_0^r \psi_z(z_b, r-\tau)\,e^{ib\tau}\tau^{-\frac{1}{2}}\,d\tau, \quad b = \frac{k_0}{2}(N_b^2-1). \tag{9}$$

In order to derive the TBC for the WAPE we consider (7) in the bottom region:

$$(\delta_b + q_1 k_0^{-2}\partial_z^2)\psi_r = i\left[\upsilon_b + (p_1-q_1)k_0^{-1}\partial_z^2\right]\psi, \quad \delta_b = 1 - q_1(1-N_b^2), \qquad z > z_b, \tag{10}$$

with $\upsilon_b = k_0\left[p_0 - 1 - (p_1-q_1)(1-N_b^2)\right]$. After a Laplace transformation of (10) in r we get

$$\left[q_1 s - i(p_1-q_1)k_0\right]\hat{\psi}_{zz}(z,s) = k_0^2(i\upsilon_b - \delta_b s)\hat{\psi}(z,s), \quad \hat{\psi}(\infty, s) = 0. \tag{11}$$

This *exterior problem* can be solved explicitly and with the matching conditions this gives

$$\hat{\psi}_z(z_b-, s) = -k_0\frac{\rho_w}{\rho_b}\sqrt[+]{\frac{i\upsilon_b - \delta_b s}{q_1 s - i(p_1-q_1)k_0}}\,\hat{\psi}(z_b-, s). \tag{12}$$

Here, $\sqrt[+]{\ }$ denotes the branch of the square root with nonnegative real part. An inverse Laplace transformation yields the TBC at the bottom for the WAPE:

$$\psi(z_b, r) = -i\eta\,\frac{\rho_b}{\rho_w}\,\psi_z(z_b, r) + \beta\,\eta\,\frac{\rho_b}{\rho_w}\int_0^r \psi_z(z_b, r-\tau)\,e^{i\theta\tau}e^{i\beta\tau}\left[J_0(\beta\tau) + i J_1(\beta\tau)\right]d\tau, \tag{13}$$

$$\eta = \frac{1}{k_0}\sqrt[+]{\frac{q_1}{\delta_b}}, \quad \beta = -\frac{p_1-p_0q_1}{2\,q_1}\,\frac{k_0}{\delta_b}, \quad \theta = \frac{p_1-q_1}{q_1}\,k_0.$$

This is a slight generalization of the TBC derived in [11] where p_0 was equal to 1.

3. DISCRETE TRANSPARENT BOUNDARY CONDITIONS

With the uniform grid points $z_j = jh$, $r_n = nk$ ($h = \Delta z$, $k = \Delta r$) and the approximation $\psi_j^n \sim \psi(z_j, r_n)$ the Crank–Nicolson difference scheme for the WAPE (7) reads:

$$\begin{aligned}\bigl[1 - q_1 V_j^{n+\frac12} + q_1 k_0^{-2} \rho_j D^0_{\frac h2}(\rho_j^{-1} D^0_{\frac h2})\bigr] D_k^+ \psi_j^n \\ = i k_0 \bigl[p_0 - 1 - (p_1 - q_1) V_j^{n+\frac12} + (p_1 - q_1) k_0^{-2} \rho_j D^0_{\frac h2}(\rho_j^{-1} D^0_{\frac h2})\bigr](\psi_j^n + \psi_j^{n+1})/2, \quad (14)\end{aligned}$$

with $V_j^{n+\frac12} := V(z_j, r_{n+\frac12})$ and $D_k^+ \psi_j^n = (\psi_j^{n+1} - \psi_j^n)/k$, $D^0_{\frac h2} \psi_j^n = (\psi^n_{j+\frac12} - \psi^n_{j-\frac12})/h$.

This scheme is second order in h and k and unconditionally stable [1]. Proceeding similarly to the derivation of (8) one can show for the discrete weighted L^2–norm

$$D_k^+ \sum_{j \in \mathbb{Z}} \frac{\bigl|\psi_j^n\bigr|^2}{\rho_j} = -C_1 k_0^{-1} \sum_{j \in \mathbb{Z}} \operatorname{Im}\Bigl\{V_j^{n+\frac12}\Bigr\} \left| \psi_j^{n+\frac12} + \frac{i q_1}{p_1 - q_1} k_0^{-1} D_k^+ \psi_j^n \right|^2 \frac{1}{\rho_j}. \qquad (15)$$

Hence, the scheme (14) preserves the norm in the dissipation–free case (V real).

In [12] Thomson and Mayfield used an *ad–hoc discretization of the analytic TBC* (9) and Mayfield showed that this *discretized TBC* for the SPE destroys the unconditional stability of the underlying Crank–Nicolson scheme [10]. Another problem of these existing discretizations is that they induce numerical reflections at the boundary, particularly when using coarse grids.

Instead of using an ad–hoc discretization of the analytic TBCs we will construct *discrete TBCs* of the fully discretized half–space problem. Our new strategy solves at no additional computational costs both problems (stability and accuracy) of the *discretized TBC*. The resulting DTBC is a generalization of the DTBC for the Schrödinger equation in [2]. The same strategy was used in [6] for advection diffusion equations.

To derive the DTBC we will now mimic the derivation of the analytic TBCs from §2 on a discrete level. For the initial data we assume $\psi_j^0 = 0$, $j \geq J-1$ and solve the discrete exterior problem in the bottom region, i.e. (14) for $j \geq J$:

$$\bigl[R\delta_b + q\Delta_h^2\bigr](\psi_j^{n+1} - \psi_j^n) = i\bigl[R\kappa_b + \Delta_h^2\bigr](\psi_j^{n+1} + \psi_j^n), \quad \delta_b = 1 - q_1(1 - N_b^2), \qquad (16)$$

with $R = 2k_0 h^2/(p_1 - q_1)/k$, $q = k q_1/(2k_0)/(p_1 - q_1)$, $\kappa_b = k k_0\bigl[p_0 - 1 - (p_1 - q_1)(1 - N_b^2)\bigr]/2$, where $\Delta_h^2 \psi_j^n = \psi_{j+1}^n - 2\,\psi_j^n + \psi_{j-1}^n$. After a Z–transformation of (16) we get

$$\bigl[z + 1 + iq(z-1)\bigr]\Delta_h^2 \hat\psi_j(z) = -iR\bigl[\delta_b(z-1) - i\kappa_b(z+1)\bigr]\hat\psi_j(z). \qquad (17)$$

The solution of (17) takes the form $\hat\psi_j(z) = \nu_1^j(z)$, $j \geq J$, where $\nu_1(z)$ solves

$$\nu^2 - 2\left[1 - \frac{iR}{2}\frac{\delta_b(z-1) - i\kappa_b(z+1)}{z + 1 + iq(z-1)}\right]\nu + 1 = 0, \quad |\nu_1(z)| < 1. \qquad (18)$$

The Z–transformed DTBC reads $\hat\psi_{J-1}(z) = \nu_1^{-1}(z)\,\hat\psi_J(z)$, and in a tedious calculation this can be inverse transformed explicitly. The DTBC for the SPE and the WAPE is:

$$(1 + iq)\,\psi_{J-1}^n = \psi_J^n * \ell_n = \sum_{m=1}^{n} \psi_J^m\,\ell_{n-m}, \quad n \geq 1, \qquad (19)$$

with the convolution coefficients $\ell_n := (1 + iq)\mathcal{Z}^{-1}\{\nu_1^{-1}(z)\}$ explicitly given in [3].

4. NUMERICAL EXAMPLE

This example appeared as the NORDA test case 3B in the PE Workshop I [8, 12]. We compare the numerical result from using our new discrete TBC to the solution using the discretized TBC for the WAPE. Due to its construction, our DTBC yields exactly (up to round–off errors) the numerical half–space solution restricted to the computational interval $[0, z_b]$. The simulation with discretized TBCs requires the same numerical effort, but their solution may (on coarse grids) strongly deviate from the half–space solution.

The environment for this example consists of an isovelocity water column ($c(z) = 1500\,\mathrm{ms}^{-1}$) over an isovelocity half–space bottom ($c_b = 1590\,\mathrm{ms}^{-1}$). The density changes at $z_b = 100\,\mathrm{m}$ from $\rho_w = 1.0\,\mathrm{gcm}^{-3}$ in the water to $\rho_b = 1.2\,\mathrm{gcm}^{-3}$ in the bottom. The source and the receiver are located at the same depth near the bottom: $z_s = z_r = 99.5\,\mathrm{m}$. The source frequency is $f = 250\,\mathrm{Hz}$. The attenuation in the water is zero, and the bottom attenuation is $\alpha_b = 0.5\,\mathrm{dB}\,/\lambda_b$, where $\lambda_b = c_b/f$ denotes the wavelength of sound in the bottom. We used the Gaussian beam from [9] as initial data. Here, the steepest angle of propagation (which is the equivalent ray–angle of the highest of the 11 propagating modes) is approximately 20° (cf. [8, 12]). Since the source is located near the bottom, the higher modes are significantly excited. Therefore the wide angle capability is important here and we use the WAPE (5) of Claerbout.

The maximum range of interest is 10 km and the reference sound speed is chosen as $c_0 = 1500\,\mathrm{ms}^{-1}$. The calculations were carried out using the depth step $\Delta z = 0.25\,\mathrm{m}$ and the range step $\Delta r = 2.5\,\mathrm{m}$. Since the source is placed close to the bottom, the TBC was applied 10 m below the ocean–bottom interface (the same was done in [12]). The typical feature of this problem is the large destructive interference null at a range of 7 km.

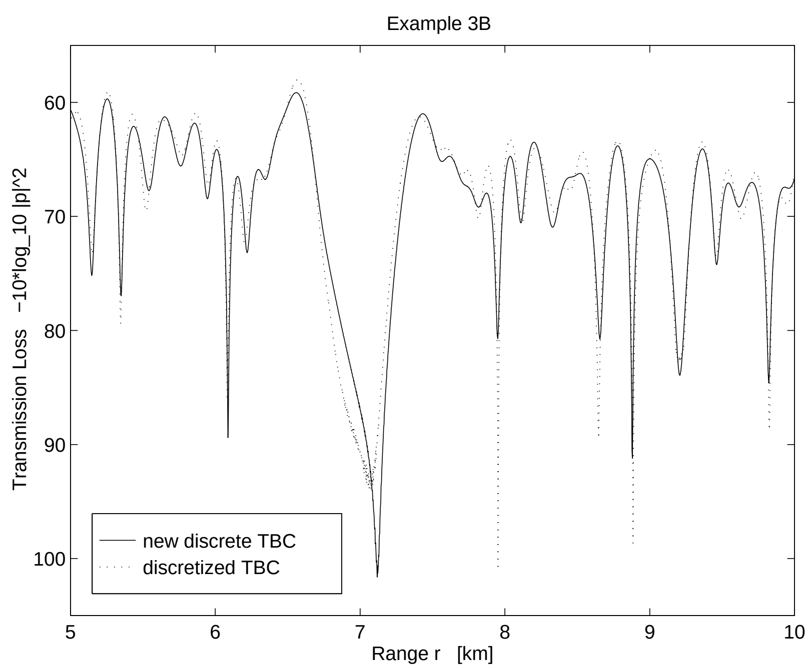

Figure: Transmission loss $-10\log_{10}|p|^2$ at $z_r = 99.5\,\mathrm{m}$ from 5 to 10 km: the solution with the new discrete TBC coincides with the half–space solution, while the solution with the discretized TBC still deviates significantly from it for the chosen discretization.

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